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Tomographic-probability description of solitons in Bose-Einstein condensates

S. De Nicola[1], R. Fedele[2,a], M.A. Man'ko[3], and V.I. Man'ko[3]

[1] Istituto di Cibernetica "Eduardo Caianiello" del CNR Comprensorio "A. Olivetti" Fabbr. 70, Via Campi Flegrei, 34, 80078 Pozzuoli (NA), Italy
[2] Dipartimento di Scienze Fisiche, Università "Federico II" di Napoli and Istituto Nazionale di Fisica Nucleare, Sezione di Napoli Complesso Universitario di M.S. Angelo, Via Cintia, 80126 Napoli, Italy
[3] P.N. Lebedev Physical Institute, Leninskii Prospect 53, Moscow 119991, Russia

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Abstract. Tomographic-probability representation for soliton solutions of Gross-Pitaevskii equation is introduced. Tomograms of the wavefunction describing bright soliton states of Bose-Einstein condensate are obtained in the presence of a quadratic external potential.

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1 Introduction

Recently, the nonlinear Schrödinger equation (NLSE) was written in the new (for nonlinear systems) tomographic probability representation [1]. This representation, associated with the linear Schrödinger equation is widely used in quantum optics (see, f.i., Ref. [2] and reference therein). When the Schrödinger equation is nonlinear (as in a number of nonlinear physical problems), its solutions can be expressed in the form of probability distribution function obeying a nonlocal nonlinear generalization of Fokker-Planck-type equation, as well. In particular, this approach can be applied to the soliton solutions of NLSE [1].

The states of Bose-Einstein condensates (BEC) are described by solutions of nonlinear Gross-Pitaevskii equation [3–5]. In comparison with the standard NLSE, this equation contains an additional linear term which depends on the potential-energy term (e.g. a harmonic-oscillator potential energy of a trap). The tomographic probability distribution map was used in [6] to write down linear von Neumann equation for density matrix [7] in the form of classical-like equation for the standard positive probability density.

The aim of our work is to obtain a combination of both the tomographic approach to NLSE developed in [1] and the approach to von Neumann equation given in [6] and to apply a generalization of this approach to nonlinear Gross-Pitaevskii equation. The solitons in BEC were observed experimentally in [8–10]. Within the framework of tomographic approach, the bright [11–13] and dark [14–18] solitons of Bose-Einstein condensate can be associated with the probability distribution functions, which describe completely the solitons in BEC. In the tomographic probability representation, the solutions to Gross-Pitaevskii equation have the form of positive probability distribution functions. This means that one can associate with solitons of BEC such characteristics as entropy, which is determined by the probability distribution and use all the mathematical tools of the probability theory.

The paper is organized as follows.

In Section 2 we review the Wigner-Moyal transform and the nonlinear von Neumann equation, while in Section 3 we present the tomographic form of linear von Neumann equation and the tomographic form of the nonlinear Schrödinger equation. In Section 4 we obtain the tomographic representation of Gross-Pitaevskii equation and discuss solitons of Bose-Einstein condensates in the tomographic representation. Some conclusions and perspectives are discussed in Section 5.

2 The Wigner-Moyal transform and the nonlinear von Neumann equation

One should point out that there exist several nonlinear generalizations of quantum mechanical evolution equations (see, for example, [19]).

Let us consider the following generalized nonlinear Schrödinger equation (NLSE):

$$i\alpha \frac{\partial \psi}{\partial s} = -\frac{\alpha^2}{2}\frac{\partial^2 \psi}{\partial x^2} + U\left[|\psi|^2, x\right]\psi, \tag{1}$$

[a] e-mail: renato.fedele@na.infn.it

where s and x are the timelike and spacelike variables, and $\psi = \psi(x,s)$ is a complex wavefunction describing the system evolution in the configuration space; $U = U\left[|\psi|^2, x\right]$ is an arbitrary real functional of $|\psi|^2$ and α is a dispersion/diffraction coefficient.

When U corresponds to a linear potential, it is well known that the system evolution can be described in the phase space x, p, where p plays the role of the conjugate momentum of x. This is done in terms of the so-called Wigner-Moyal transform, introduced first by Wigner [20] and later by Moyal [21], which is defined as (usually also referred as to Wigner-Weyl map):

$$W_\psi(x,p,s) = \frac{1}{2\pi\alpha}\int \rho_\psi\left(x+\frac{u}{2}, x-\frac{u}{2}, s\right) e^{\frac{i}{\alpha}pu}\, du, \quad (2)$$

where ρ is the density matrix defined as $\rho(x', x'', s) = \psi^*(x', s)\psi(x'', s)$.

The inverse of the Fourier transform (2) reads

$$\psi(x,s)\psi^*(x',s) = \int W_\psi\left(\frac{x+x'}{2}, p, s\right) e^{\frac{i}{\alpha}p(x-x')}\, dp. \quad (3)$$

In particular, from (3) we have

$$|\psi(x,s)|^2 = \int W_\psi\left(x,p,s\right)\, dp, \quad (4)$$

and

$$\psi^*(0,s)\psi(x,s) = \int W_\psi\left(\frac{x}{2}, p, s\right) e^{\frac{i}{\alpha}px}\, dp. \quad (5)$$

Suppose that $\psi(0,s)$ is not equal to zero. Thus, for pure states, given Wigner function one can reconstruct the complex wave function up to a time-phase factor.

It can be shown that W_ψ satisfies an equation which is the analog of the Liouville equation

$$\frac{\partial W_\psi}{\partial s} + p\frac{\partial W_\psi}{\partial x} = \frac{i}{2\pi\alpha^2}\int dy \int dp' \,\exp\left\{i\frac{y}{\alpha}(p-p')\right\} \times\left\{U\left(x+\frac{y}{2}\right) - U\left(x-\frac{y}{2}\right)\right\} W_\psi(x,p',s), \quad (6)$$

which can be cast as the following von Neumann-Weyl equation

$$\frac{\partial W_\psi}{\partial s} + p\frac{\partial W_\psi}{\partial x} + \frac{i}{\alpha}\left\{U\left(x+\frac{i\alpha}{2}\frac{\partial}{\partial p}\right) - U\left(x-\frac{i\alpha}{2}\frac{\partial}{\partial p}\right)\right\} W_\psi(x,p',s) = 0. \quad (7)$$

When U is a nonlinear potential, it is easy to prove that (6) and (7) can be still valid and in particular they become

$$\frac{\partial W_\psi}{\partial s} + p\frac{\partial W_\psi}{\partial x} = \frac{i}{2\pi\alpha^2}\int dy \int dp' \,\exp\left\{i\frac{y}{\alpha}(p-p')\right\} \times\left\{U\left[\left|\psi\left(x+\frac{y}{2}, s\right)\right|^2\right] - U\left[\left|\psi\left(x-\frac{y}{2}, s\right)\right|^2\right]\right\} \times W_\psi(x,p',s) \quad (8)$$

or

$$\frac{\partial W_\psi}{\partial s} + p\frac{\partial W_\psi}{\partial x} + \frac{i}{\alpha}\left\{U\left[\left|\psi\left(x+\frac{i\alpha}{2}\frac{\partial}{\partial p}, s\right)\right|^2\right] - U\left[\left|\psi\left(x-\frac{i\alpha}{2}\frac{\partial}{\partial p}, s\right)\right|^2\right]\right\} W_\psi(x,p,s) = 0. \quad (9)$$

The nonlinear von Neumann equation (9) has been recently used to describe the nonlinear dynamics of an electromagnetic wavepacket through a nonlinear Kerr medium with a memory nonlinear term [22] as well as the longitudinal nonlinear collective dynamics of a charged-particle bunch, within the framework of the Thermal Wave Model, in circular accelerating machines with both reactive (accounting for a Kerr-like nonlinear term) and resistive (accounting for a memory-like nonlinear term) parts of the coupling impedance [23]. In these studies, the wavepacket/bunch have been considered as described by the analog of a quantum pure state. A careful phase-space modulational instability analysis, showing the existence of a Landau-type damping of the wavepacket/bunch has been carried out [22,23]. In an approximate approach, which takes into account a sort of analog of the quantum mixed states, a phase-space modulational instability analysis (including the effects of the Landau-type damping) of an ensemble of large-amplitude partially incoherent waves has been carried out, as well, for surface gravity waves in the ocean physics [24], for electromagnetic waves in Kerr media [25] and for Langmuir envelopes in plasma physics [26].

By virtue of (4), equations (8) and (7) become the following nonlinear integro-differential equation (nonlinear von Neumann-Weyl equation)

$$\frac{\partial W_\psi}{\partial s} + p\frac{\partial W_\psi}{\partial x} = \frac{i}{2\pi\alpha^2}\int dy \int dp' \,\exp\left\{i\frac{y}{\alpha}(p-p')\right\} \times\left\{U\left[\int W_\psi\left(x+\frac{y}{2}, p, s\right) dp\right] - U\left[\int W_\psi\left(x-\frac{y}{2}, p, s\right) dp\right]\right\} W_\psi(x,p',s) \quad (10)$$

or

$$\frac{\partial W_\psi}{\partial s} + p\frac{\partial W_\psi}{\partial x} + \frac{i}{\alpha}\left\{\, U\left[\int W_\psi\left(x+\frac{i\alpha}{2}\frac{\partial}{\partial p}, p, s\right) dp\right] - U\left[\int W_\psi\left(x-\frac{i\alpha}{2}\frac{\partial}{\partial p}, p, s\right) dp\right]\,\right\} W_\psi(x,p,s) = 0. \quad (11)$$

Equation (10) or (11) describes the phase-space evolution of the system which, in the configuration space, is governed by the NLSE (1).

3 Tomographic form of (nonlinear) von Neumann equation

Following [1,2,6,27] in this section we describe the tomographic map of arbitrary complex function $\psi(x,s)$

(a wave function of BEC) onto nonnegative probability distribution function $w(X,\mu,\nu)$ called tomogram. The map is given by the relation

$$w(X,\mu,\nu,s) = \frac{1}{2\pi|\nu|\alpha}\left|\int \psi(y,s)\exp\left(\frac{i\mu}{2\nu\alpha}\,y^2-\frac{iX}{\nu\alpha}\,y\right)dy\right|^2. \quad (12)$$

One can see that the tomogram has homogeneity property

$$w(\lambda X,\lambda\mu,\lambda\nu,s)=\frac{1}{|\lambda|}w(X,\mu,\nu,s). \quad (13)$$

For the normalized function $\psi(x,s)$, the tomogram satisfies the normalization condition

$$\int w(X,\mu,\nu,s)\,dX=1. \quad (14)$$

The tomogram has the physical meaning of probability distribution function of random variable X. This probability distribution depends on two real parameters μ and ν. These parameters label reference frame in the phase space where the position coordinate X is considered [6]. Using the real Wigner function [20]

$$W_\psi(q,p,s)=\frac{1}{2\pi\alpha}\int \psi^*\Big(q+\frac{u}{2},s\Big)\psi\Big(q-\frac{u}{2},s\Big)e^{\frac{i}{\alpha}pu}\,du, \quad (15)$$

one can find, for any s, the tomogram in the form

$$w(X,\mu,\nu,s)=\int W_\psi(q,p,s)\delta(X-\mu q-\nu p)\,dq\,dp\,. \quad (16)$$

The Wigner function can be reconstructed if one knows the tomogram

$$W_\psi(q,p,s) = \int w(X,\mu,\nu,s)\exp\left[i(X-\mu q-\nu p)\right]\frac{dX\,d\mu\,d\nu}{(2\pi)^2}. \quad (17)$$

Using the relations written above one can prove that there exist the following correspondence rules [1,28,29]:

$$\begin{aligned}
&\psi(x,s)\psi^*(x',s)\longrightarrow w(X,\mu,\nu,s);\\
&x\psi(x,s)\psi^*(x',s)\longrightarrow\left[-\left(\frac{\partial}{\partial X}\right)^{-1}\frac{\partial}{\partial\mu}-\frac{i}{2}\,\nu\,\frac{\partial}{\partial X}\right]\\
&\quad\times w(X,\mu,\nu,s);\\
&\frac{\partial}{\partial x}\psi(x,s)\psi^*(x',s)\longrightarrow\left[\frac{\mu}{2}\,\frac{\partial}{\partial X}+i\left(\frac{\partial}{\partial X}\right)^{-1}\frac{\partial}{\partial\nu}\right]\\
&\quad\times w(X,\mu,\nu,s);\\
&x'\psi(x,s)\psi^*(x',s)\longrightarrow\left[-\left(\frac{\partial}{\partial X}\right)^{-1}\frac{\partial}{\partial\mu}+\frac{i}{2}\,\nu\,\frac{\partial}{\partial X}\right]\\
&\quad\times w(X,\mu,\nu,s);\\
&\frac{\partial}{\partial x'}\psi(x,s)\psi^*(x',s)\longrightarrow\left[\frac{\mu}{2}\,\frac{\partial}{\partial X}-i\left(\frac{\partial}{\partial X}\right)^{-1}\frac{\partial}{\partial\nu}\right]\\
&\quad\times w(X,\mu,\nu,s),
\end{aligned} \quad (18)$$

which can be used to obtain the differential equation for tomogram from known differential equation for the wave function of BEC. Analogous correspondence rules can be obtained to construct the equation for Wigner function of BEC state.

According to the results given in Section 2 the evolution equation for Wigner function, in both linear and nonlinear cases, can be cast in the following form:

$$\frac{\partial W(q,p,s)}{\partial s}+p\,\frac{\partial W(q,p,s)}{\partial q} -2\,\mathrm{Im}\left[U\left(q+\frac{i}{2}\frac{\partial}{\partial p}\right)\right]W(q,p,s)=0. \quad (19)$$

Thus, the correspondence rules (18) give the tomographic form of the von Neumann equation

$$\frac{\partial w(X,\mu,\nu,s)}{\partial s}+\mu\,\frac{\partial w(X,\mu,\nu,s)}{\partial \nu} -2\,\mathrm{Im}\,U\left[\left(\frac{\partial}{\partial X}\right)^{-1}\frac{\partial}{\partial\mu}+\frac{i}{2}\,\nu\,\frac{\partial}{\partial X}\right]w(X,\mu,\nu,s)=0. \quad (20)$$

In view of the physical meaning of the tomogram as the probability distribution, one can introduce the entropy associated to solution of equation (20)

$$S(\mu,\nu,s)=-\int w(X,\mu,\nu,s)\ln w(X,\mu,\nu,s)\,dX. \quad (21)$$

This entropy is an additional characteristic of the solution of the evolution equation.

4 Solitons of BEC in tomographic representation

In this section, we consider the 3D Gross-Pitaevskii equation describing mean field of the BEC (see, f.i., [30]):

$$\left\{-\frac{\hbar^2}{2m}\nabla^2+gN|\psi(\boldsymbol{r},t)|^2+\frac{1}{2}m\left[\omega_\perp^2(x^2+y^2)+\omega_z^2z^2\right]\right\} \times\psi(\boldsymbol{r},t)=i\hbar\frac{\partial}{\partial t}\psi(\boldsymbol{r},t), \quad (22)$$

where $g=4\pi^2a/m$, a is the 's-wave' scattering length, m is the atomic mass, N is the number of atoms in the condensate, and $\omega_\perp$ and ω_z are axial and longitudinal oscillation frequencies of the atoms in the trapping potential, respectively.

To look for a normalized stationary solution of equation (22), we write

$$\psi(\boldsymbol{r},t)\equiv\psi_a(\boldsymbol{r})\exp\left\{-\frac{i}{\hbar}E_at\right\}.$$

Thus, in the context of the variational method of solving the stationary GP equation, the Gross-Pitaevskii energy-functional, defined as

$$E_{GP}[\psi_a] \equiv \left\langle \psi_a \left| \hat{H} \right| \psi_a \right\rangle,$$

has to be extremized ($\hat{H}$ being the Hamiltonian operator associated with Eq. (22)). An approximate variational solution of the stationary GP equation, in the form of a bright soliton, has been recently shown [13], i.e.

$$\psi_a(\boldsymbol{r}) = \frac{1}{\sqrt{2\pi\sigma_\perp^2 \ell_z}} \exp\left(-\frac{x^2+y^2}{2\sigma_\perp^2}\right) \operatorname{sech}\left(\frac{z}{\ell_z}\right). \quad (23)$$

This solution is important for explosive potential for which $\omega_z^2 < 0$. Here $\sigma_\perp$ and ℓ_z are the variational parameters, which describe the transverse and axial widths of the wave function [13]. Notice that solution (23) is factorized into two terms depending on the transverse (x, y) and longitudinal (z) coordinates. This allows us to use the tomographic approach developed in the previous sections to analyze the longitudinally dependent part of the solution (23). Indeed, the quasi-1D limit of the 3D Gross-Pitaevskii equation gives the following equation [13]:

$$\left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial z^2} + g_{1D}N|\phi|^2 + \hbar\omega_\perp + \frac{1}{2}m\omega_z^2 z^2\right]\phi = i\hbar\frac{\partial \phi}{\partial t}, \quad (24)$$

where $g_{1D} = 2a\hbar\omega_\perp$ is the renormalized quasi-1D coupling constant. The normalized solution $\phi(z,t)$ for the 1D Gross-Pitaeviskii equation can be written in the form of bright-soliton-like wave, namely,

$$\phi(z,t) = \frac{1}{\sqrt{2\ell_z}} \operatorname{sech}\left(\frac{z}{\ell_z}\right) \exp\left(-\frac{i\mu_0 t}{\hbar}\right), \quad (25)$$

where

$$\mu_0 = \hbar\omega_z - N^2 g_{1D}^2 m/(8\hbar^2)$$

is the chemical potential and

$$l_z = 2\hbar^2/(m|g_{1D}|N).$$

The z-dependent part is essentially the longitudinal factor of the guessed variational ansatz for the stationary solution (23). To determine the tomographic probability distribution of the longitudinal motion let us use our equation (12) and let $\alpha = 1$. Now, the complex function $\psi(y,t)$ is given in the above notation by equation (25), with the substitution $z \to yL$, where $L = \sqrt{\hbar/m\omega_\perp}$ is the normalization length for the 1D Gross-Pitaevskii equation (24) can be cast in the following dimensionless form:

$$i\frac{\partial \phi}{\partial \tau} = -\frac{1}{2}\frac{\partial^2 \phi}{\partial y^2} + U\left[|\phi|^2, y\right]\phi, \quad (26)$$

where $\tau = \omega_\perp t$ and

$$U[|\phi|^2, y] = 2aN|\phi|^2 + \frac{1}{2}\left(\frac{\omega_z}{\omega_\perp}\right)^2 y^2.$$

It should be pointed out that the soliton solution under consideration is an approximate solution of the initial 3D Gross-Pitaevskii equation.

The tomographic probability distribution $w(X, \mu, \nu)$ can be cast in the following dimensionless form:

$$\tilde{w}(X,\mu,\nu) \equiv Lw(X,\mu,\nu) = \frac{1}{2\pi|\nu|} \times \left|\int \sqrt{\frac{\gamma}{2}} \operatorname{sech}(\gamma y) \exp\left(\frac{i\mu}{2\nu}y^2 - \frac{iX}{\nu}y\right) dy\right|^2, \quad (27)$$

where the dependence on the parameter

$$\gamma = L/\ell_z = Lmg_{1D}N/(2\hbar^2)$$

governing the width of the longitudinal soliton distribution is shown. Note that the time independence of the tomogram (27) is due to the special form of solution (25). In fact, once (25) is substituted in (12), the time-dependent exponential factor automatically disappears.

To illustrate the behaviour of the tomogram, we take $\mu = \cos\theta$, $\nu = \sin\theta$.

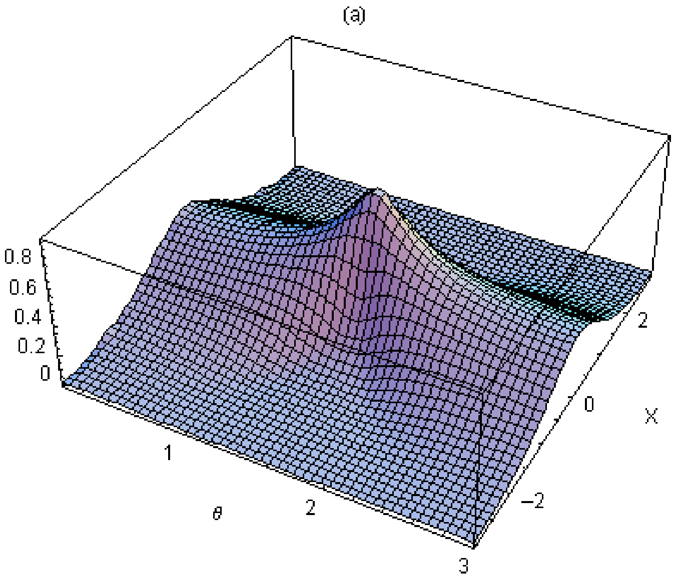
The 3D plot of the tomogram $\tilde{w}(X, \cos\theta, \sin\theta)$ of the bright-soliton-like solution is displayed in the X-θ plane in Figure 1a, while the corresponding density plot is shown in Figure 1b. The spread of the tomographic map is basically governed by the dimensionless parameter $\gamma = L/\ell_z$. The value of $\gamma \approx 0.82$ in Figure 1 is adopted from experimental conditions reported in [31], where $L = 1.4$ μm and the axial width distribution $\ell_z = 1.7$ μm.

5 Conclusions

To conclude, we summarize the main result of the paper.

We obtained the probability description of BEC states. It means that we mapped, e.g., the BEC solitons onto probability distribution functions, which contain the complete information on the solitons. It is quite unexpected result that arbitrary state of BEC can be associated to the standard probability distribution and, since the tomographic map employed is invertable, the space-time dependence of the soliton can be reconstructed (up to the phase factor) if one knows the introduced tomographic probability distribution of the soliton.

We get the tomographic probability form of Gross-Pitaevskii equation as well as Moyal form of this equation. The equation in the probability representation can be considered as a nonlinear analog of the classical Fokker-Planck equation for classical stochastic process. It is worthy to point out that the nonlinear dynamical equations like nonlinear Schrödinger equation and, in particular, the Gross-Pitaevskii equation, can be rewritten (though in a more complicated form) for the standard positive probability distribution obeying to analogs of classical nonlinear Fokker-Planck-type equations. Up to our knowledge, equations of this kind have not been studied in classical



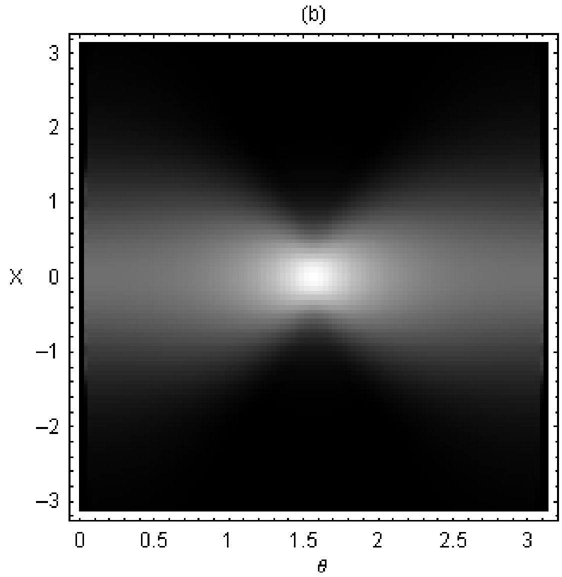


Fig. 1. (a) Tomogram of the bright soliton-like solution as function of X and θ; (b) Density plot in the (X,θ) plane. For both plots $\gamma = L/\ell_z \approx 0.82$ ($L = 1.4$ μm, $\ell_z = 1.7$ μm), according to the BEC experimental conditions reported in [31].

statistics. Also the Moyal form of the nonlinear dynamical equations is another new aspect of BEC dynamics discussed in our paper. A potential usefulness of the suggested probability representation for BEC solitons is the possibility to apply well-elaborated theorems of probability theory to study such properties as propagation and asymptotics of the probabilities in more complicated situations.

We have studied tomograms of specific soliton solution for the Gross-Pitaevskii equation.

One can extend the tomographic and Moyal descriptions also to other types of BEC states like kink states studied in [32].

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